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Shock Structure in the Mixture of Gases: Stability and Bifurcation of Equilibria

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1. Introduction

List of symbols

$\mathbf{A}^0, \mathbf{A}$	Jacobian matrices	$\mathbf{t}, \mathbf{t}_\alpha, \mathbf{t}_b$	stress tensor
$\mathbf{B}$	viscosity matrix	$\mathbf{u}, \mathbf{U}, \mathbf{v}$	state variables
c, c_α, c_b	concentration	u	relative velocity
c_S, c_0	local speed of sound	$\mathbf{u}_\alpha, \mathbf{u}_b$	diffusion velocity
e	internal energy density	$v, \mathbf{v}, \mathbf{v}_\alpha, \mathbf{v}_b$	velocity
$e_\alpha, e_b, \hat{e}_b$	energy density source term	x	space variable
$\mathbf{F}^0, \hat{\mathbf{F}}^0, \mathbf{f}^0, \mathbf{g}^0$	densities	*Greek symbols*	
$\mathbf{F}, \hat{\mathbf{F}}, \mathbf{f}, \mathbf{g}$	fluxes	γ	ratio of specific heats
$\mathbf{h}$	equilibrium manifold	ε	small parameter
$\mathbf{J}$	diffusion flux	$\varepsilon, \varepsilon_I, \varepsilon_\alpha, \varepsilon_b$	internal energy density
k_B	Boltzmann constant	$\varphi_{bc}, \theta_{bc}, \psi_{bc}$	phenomenological coefficients
l_0	reference length		
m	atomic mass	λ_i, Λ_j	eigenvalues
$\mathbf{m}_\alpha, \mathbf{m}_b, \hat{\mathbf{m}}_b$	momentum density source term	κ	heat conductivity
		Θ	temperature difference
μ	viscosity	$\rho, \rho_\alpha, \rho_b$	mass density
M_0	Mach number	σ	stress
n, N	number of state variables	Σ	singular surface
p, p_α	pressure	$\tau, \tau_\sigma, \tau_q$	relaxation time
$q, \mathbf{q}, \mathbf{q}_\alpha, \mathbf{q}_b$	heat flux	$\tau_\alpha, \tau_b, \hat{\tau}_b$	mass density source term
$\mathbf{q}, \mathbf{Q}$	source terms	ξ	lumped space variable
$\mathbf{r}_i, \mathbf{R}_j$	eigenvectors	*Subscripts*	
s	shock speed	E	equilibrium
t	time variable	0, 1	upstream, downstream
T, T_α, T_b	temperature		

Shock waves are moving singular surfaces on which jump discontinuities of field variables occur. Such singularities are commonly related to mathematical models in the form of

hyperbolic systems of conservation laws. However, in real physical systems dissipative mechanisms of thermo-mechanical nature smear out the discontinuity and transform the shock wave into a *shock structure* – travelling wave with large gradients of field variables in the neighborhood of singular surface. Occurrence of dissipation ought to be reflected on the mathematical model. Actually, there are two common types of dissipative models: (i) a parabolic one which takes the dissipative effects into account through diffusion-like terms, and (ii) a hyperbolic system of balance laws which extends the original system with a set of hyperbolic balance laws which comprise relaxation-type source terms. Both of these models strive to capture effects which push the system out of local equilibrium state, although the mathematical structure is rather different. This chapter gives a review of recent results about the shock structure in dissipative hyperbolic systems of balance laws and provides insight into their application to recently developed hyperbolic model of gaseous mixtures.

Study of shock structure usually starts with travelling wave ansatz which transforms the original system of partial differential equations (PDE's) to a system of ordinary differential equations (ODE's). It was pointed out in a series of papers (Simić, 2008; 2009) that appearance of physically admissible shock structure can be anticipated by stability and bifurcation analysis of stationary points of the ODE system. It was recognized that these stationary points are related by Rankine-Hugoniot equations for the equilibrium subsystem which is obtained when dissipation (relaxation) is neglected. However, these results can be derived without prior knowledge of Rankine-Hugoniot conditions on singular surface and without explicit use of selection rules for physically admissible shock waves, like Lax, Liu or entropy growth condition.

Stability and bifurcation results can be easily predicted in the case of parabolic systems (with diffusion-like dissipation) since equilibrium subsystem is naturally imbedded into its structure. On the other hand, hyperbolic dissipative systems (with relaxation-type source terms) absorb the equilibrium subsystem into its structure, but also make its properties less visible. So far, the shock structure is analyzed in dissipative Burgers equation, isothermal viscoelasticity (Simić, 2008) and continuum models of gas dynamics (viscous and heat-conducting gases) (Simić, 2009). In all these models dissipation can be included in either way. Our aim is to show that methodology used in the above mentioned examples can be efficiently applied also to a rather complex hyperbolic dissipative system which models the multi-temperature mixture of gases. Thus far this model does not have its parabolic counterpart. In particular, the shock structure problem will be studied via stability and bifurcation analysis in the case of binary mixture.

The chapter will be structured as follows. First, the general framework will be given, which includes the review of dissipative mathematical models and an overview of the shock structure problem. In the sequel a review of recent results about the shock structure problem in hyperbolic systems will be given. They comprise the isothermal viscoelasticity model and continuum model of gas dynamics. Finally, the shock structure problem will be discussed in the case of binary multi-temperature mixture, a new issue which will be studied for the first time in this context.

2. General framework

2.1 The mathematical structure of dissipative models

To make the ideas more clear, a brief review of mathematical structure of dissipative models will be given. For the sake of simplicity we shall confine the exposition to one space dimension indicated by variable x.

It is well known that continuum models in physics are usually expressed in the form of conservation laws

$$\partial_t \mathbf{F}^0(\mathbf{u}) + \partial_x \mathbf{F}(\mathbf{u}) = \mathbf{0} \tag{1}$$

where $\mathbf{u}(x,t) \in \mathbf{R}^n$ is the vector of state variables, $\mathbf{F}^0(\mathbf{u})$ is the vector of densities and $\mathbf{F}(\mathbf{u})$ the vector of fluxes. When densities and fluxes are functions of state variables solely, i.e. not of their derivatives, it is expected that (1) is hyperbolic, at least in some region of the state space. In other words, the eigenvalue problem

$$\begin{aligned} &(-\lambda \mathbf{A}^0(\mathbf{u}) + \mathbf{A}(\mathbf{u}))\mathbf{r} = \mathbf{0}; \\ &\mathbf{A}^0(\mathbf{u}) = \partial \mathbf{F}^0(\mathbf{u})/\partial \mathbf{u}; \qquad \mathbf{A}(\mathbf{u}) = \partial \mathbf{F}(\mathbf{u})/\partial \mathbf{u}, \end{aligned} \tag{2}$$

has n real eigenvalues $\lambda_i(\mathbf{u})$ called characteristic speeds, and n linearly independent eigenvectors $\mathbf{r}_i(\mathbf{u})$, $i = 1, \ldots, n$; $\mathbf{A}^0(\mathbf{u})$ is assumed to be nonsingular. This property permits modelling of wave propagation through space with finite speeds.

Hyperbolicity is the main cause for non-existence of smooth solutions for all t, even when initial data are smooth. Jump discontinuities – shock waves – which are located on the singular surface $\Sigma(x,t)$, may appear in finite time interval. If $[[\cdot]] = (\cdot)_1 - (\cdot)_0$ denotes the jump of any quantity in front (upstream) $(\cdot)_0$ and behind (downstream) $(\cdot)_1$ the surface Σ which propagates with speed s, then Rankine-Hugoniot conditions relate the jump of the field variables to the shock speed

$$[[\mathbf{F}(\mathbf{u})]] = s[[\mathbf{F}^0(\mathbf{u})]]. \tag{3}$$

However, not all of the jump discontinuities which satisfy (3) are observable in reality. Along with Rankine-Hugoniot conditions, they have to satisfy additional ones which serve as selection rules for physically admissible solutions (Dafermos, 2000; Serre, 1999). For the purpose of this exposition we shall be confined with the Lax condition

$$\lambda_i(\mathbf{u}_1) \geq s \geq \lambda_i(\mathbf{u}_0). \tag{4}$$

It represents a particular form of irreversibility condition since the roles of $\mathbf{u}_0$ and $\mathbf{u}_1$ cannot be interchanged. In classical gas dynamics it is related to the fact that physically admissible shock wave has to be supersonic with respect to upstream state and subsonic with respect to downstream state.

Classical continuum thermomechanics motivates so-called viscosity approach which regularizes the system (1) by adding parabolic terms

$$\partial_t \mathbf{F}^0(\mathbf{u}) + \partial_x \mathbf{F}(\mathbf{u}) = \varepsilon \partial_x (\mathbf{B}(\mathbf{u}) \partial_x \mathbf{u}) \tag{5}$$

where $\mathbf{B}(\mathbf{u})$ is viscosity matrix and $\varepsilon > 0$ small parameter, usually related to viscosity, heat conductivity and diffusivity. This model predicts infinite speed of propagation of disturbances, and characteristic speeds, as one of the basic features of hyperbolic model (1), cannot be related to (5). However, parabolic terms regularize the shock waves which correspond to genuinely nonlinear characteristic speeds of (1). In such a way (5) comprise solutions representing the shock profile travelling uniformly with shock speed s. These models are proved to be reliable as long as the state of the system is not far from the local equilibrium one.

Another way of description of dissipative mechanisms is to take into account the relaxation effects. Formally, this assumes extension of the set of state variables $\mathbf{u} \in \mathbf{R}^n$ by $\mathbf{v} \in \mathbf{R}^k$, $n + k = N$, which are governed by the additional set of balance laws. In particular, we have

$$\partial_t \hat{\mathbf{F}}^0(\mathbf{U}) + \partial_x \hat{\mathbf{F}}(\mathbf{U}) = \frac{1}{\tau}\mathbf{Q}(\mathbf{U}), \tag{6}$$

where

$$\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}, \qquad \mathbf{Q}(\mathbf{U}) = \begin{pmatrix} \mathbf{0} \\ \mathbf{q}(\mathbf{u},\mathbf{v}) \end{pmatrix}, \tag{7}$$
$$\hat{\mathbf{F}}^0(\mathbf{U}) = \begin{pmatrix} \mathbf{f}^0(\mathbf{u},\mathbf{v}) \\ \mathbf{g}^0(\mathbf{u},\mathbf{v}) \end{pmatrix}, \qquad \hat{\mathbf{F}}(\mathbf{U}) = \begin{pmatrix} \mathbf{f}(\mathbf{u},\mathbf{v}) \\ \mathbf{g}(\mathbf{u},\mathbf{v}) \end{pmatrix},$$

and $\tau > 0$ is a small parameter – relaxation time. It is assumed that $\mathbf{q}(\mathbf{u},\mathbf{v}) = \mathbf{0}$ uniquely determines the "equilibrium manifold" $\mathbf{v}_{\mathrm{E}} = \mathbf{h}(\mathbf{u})$ as $\tau \to 0$, on which the system (6) reduces to (1) with $\mathbf{F}^0(\mathbf{u}) = \mathbf{f}^0(\mathbf{u},\mathbf{h}(\mathbf{u}))$ and $\mathbf{F}(\mathbf{u}) = \mathbf{f}(\mathbf{u},\mathbf{h}(\mathbf{u}))$. The first n equations in (6) are conservation laws, while remaining k ones are balance laws with source terms $\mathbf{q}(\mathbf{u},\mathbf{v})/\tau$ which describe dissipative effects off the equilibrium manifold.
It is customary to expect that system (6) is hyperbolic at least in some subset of the extended state space $\mathbf{R}^N$ which contains the equilibrium manifold. Corresponding characteristic speeds $\Lambda_j(\mathbf{U})$, $j = 1,\ldots,N$, and the set of linearly independent eigenvectors $\mathbf{R}_j(\mathbf{U})$ are determined from the eigenvalue problem for the differential part of (6)

$$(-\Lambda\hat{\mathbf{A}}^0(\mathbf{U}) + \hat{\mathbf{A}}(\mathbf{U}))\mathbf{R} = \mathbf{0}; \tag{8}$$
$$\hat{\mathbf{A}}^0(\mathbf{U}) = \partial\hat{\mathbf{F}}^0(\mathbf{U})/\partial\mathbf{U}; \qquad \hat{\mathbf{A}}(\mathbf{U}) = \partial\hat{\mathbf{F}}(\mathbf{U})/\partial\mathbf{U}.$$

Important property of characteristic speeds $\Lambda_j(\mathbf{U})$ is that they provide bounds for the characteristic speeds of (1) on equilibrium manifold through the subcharacteristic condition (Boillat & Ruggeri, 1997; Chen et al., 1994)

$$\min_{1\le j\le N} \Lambda_j(\mathbf{u},\mathbf{h}(\mathbf{u})) \le \lambda_i(\mathbf{u}) \le \max_{1\le j\le N} \Lambda_j(\mathbf{u},\mathbf{h}(\mathbf{u})). \tag{9}$$

However, the spectrum $\lambda_i(\mathbf{u})$ of the system (1) does not have to be contained in the spectrum $\Lambda_j(\mathbf{u},\mathbf{h}(\mathbf{u}))$ of the hyperbolic dissipative system (6), i.e. λ's may not coincide with Λ's on the equilibrium manifold. Moreover, Rankine-Hugoniot conditions for (6) have different form than for (1), and consequently may predict jump discontinuities which appear off the equilibrium manifold. These discrepancies between hyperbolic systems and their dissipative counterparts call for the answer to the question how can one relate the jump discontinuities of (1) to the shock structure solutions expected to be derived from (6). First results may be found in (Liu, 1987) for systems of two balance laws. General existence result for shock profiles has been given in (Yong & Zumbrun, 2000), under certain reasonable structural conditions.
It is important to note that dissipative character of parabolic (5) and hyperbolic (6) system comes from the compatibility with entropy inequality. In former case it imposes restrictions on viscosity matrix $\mathbf{B}(\mathbf{u})$, while in the latter one determines the structure of source terms $\mathbf{q}(\mathbf{u},\mathbf{v})$. In parabolic case Coleman-Noll procedure is usually applied (de Groot & Mazur, 1984). For hyperbolic case a systematic procedure is developed within the framework of extended thermodynamics (Müller & Ruggeri, 1998), based upon exploitation of Liu's method of multipliers (Liu, 1972) and generalized form of entropy flux.

2.2 Shock structure problem and stability analysis

The shock structure problem has long and rich history. First attempt to apply dynamical systems theory in the study of shock structure was (Gilbarg & Paolucci, 1953). This approach provided an excellent framework which persisted and still remains one of the basic tools in complex problems, e.g. shock waves in higher-order approximations (Holian et al., 2011; Uribe et al., 2000), predictions using modified Mott-Smith method (Solovchuk & Sheu, 2011) and generalizations of Navier-Stokes equations (Uribe, 2011).

However, stability and bifurcation analysis related to the shock structure problem had been mostly applied in the study of Boltzmann equation. Nicolaenko and Thurber (Nicolaenko & Thurber, 1975) showed that Boltzmann operator for hard sphere potential has nontrivial eigenvalue which changes the sign in transition from subsonic to supersonic regimes. Caflisch and Nicolaenko (Caflisch & Nicolaenko, 1982) generalized these results for hard cut-off potentials and proved the existence of shock profile solutions. Existence results for discrete velocity models of the Boltzmann equation, which also have a flavor of bifurcation theory, have been proven in (Bose et al., 1998) and (Bernhoff & Bobylev, 2007). Recently, center manifold reduction has been applied to prove the existence of weak shocks for the classical parabolic model of gas dynamics and bifurcation equation of transcritical type appeared there as a side result (Lorin, 2003).

This problem was also extensively studied within the framework of extended thermodynamics (Weiss, 1995), whose models are strongly related to ones obtained by the method of moments in kinetic theory of gases (Grad, 1949). One of the most important shortcomings of these models, recognized at the early stage (Grad, 1952), is that continuous shock structure seize to exist when shock speed exceeds certain critical value. It was proven (Boillat & Ruggeri, 1998) that this is an intrinsic property of hyperbolic systems of balance laws endowed with convex entropy, and the critical value of shock speed coincides with the highest characteristic speed. Since our study will be focused on shock speeds in the neighborhood of the highest characteristic speed of equilibrium subsystem, rather than complete hyperbolic system, we shall not be confronted with this problem.

3. Isothermal viscoelasticity

In this section we shall analyze the shock structure problem in a simple mathematical model of isothermal viscoelasticity. It will give us an idea of the method to be used in more involved situations and provide a flavor of basic results concerned with stability and bifurcation of equilibria in this problem. Underlying equilibrium system will be the so-called $p-$system

$$\begin{aligned} \partial_t u^1 - \partial_x u^2 &= 0, \\ \partial_t u^2 - \partial_x p(u^1) &= 0. \end{aligned} \tag{10}$$

Although primarily arose in the context of isothermal elastodynamics, it is paradigmatic model for nonlinear wave propagation. The system is hyperbolic provided $p'(u^1) > 0$ with characteristic speeds

$$\lambda_1(\mathbf{u}) = -\sqrt{p'(u^1)}, \quad \lambda_2(\mathbf{u}) = \sqrt{p'(u^1)}, \tag{11}$$

for the vector of equilibrium variables $\mathbf{u} = (u^1, u^2)^T$. We shall assume that eigenvalues (11) are genuinely nonlinear (see Dafermos (2000)), which implies $p''(u^1) \neq 0$ for all u^1. Our analysis will be focused on weak shocks which appear as bifurcating solution in the neighborhood of

the highest characteristic speed, i.e. when $s \to s^* = \lambda_2(\mathbf{u}_0)$ and satisfy Lax condition

$$\lambda_2(\mathbf{u}_0) < s < \lambda_2(\mathbf{u}_1).$$

They are determined as solutions of Rankine-Hugoniot equations

$$\begin{aligned} s\left(u_0^1 - u_1^1\right) + \left(u_0^2 - u_1^2\right) &= 0; \\ s\left(u_0^2 - u_1^2\right) + \left(p(u_0^1) - p(u_1^1)\right) &= 0. \end{aligned} \tag{12}$$

A simple way to capture nonequilibrium and dissipative effects is to extend the set of field variables by introducing nonequilibrium ones. Their evolution is governed by additional balance laws of relaxation type. In the case of $p-$system such a model has been proposed in (Suliciu, 1990)

$$\begin{aligned} &\partial_t u^1 - \partial_x u^2 = 0, \\ &\partial_t u^2 - \partial_x v = 0, \\ &\partial_t v - \nu \partial_x u^2 = -\frac{1}{\varepsilon}\left(v - p(u^1)\right), \end{aligned} \tag{13}$$

where $\nu > 0$ and ε is small positive parameter which plays the role of relaxation time. Its aim was to describe isothermal viscoelastic response of a continuum. Heuristically, the difference $v - p(u^1)$ acts as a driving agent which pushes the system towards local equilibrium state $v = p(u^1)$. On the other hand, the smaller ε is, the more rapid will be the response of the material. As a matter of fact, when $\varepsilon \to 0$ an *equilibrium manifold* $v_{\mathrm{E}} = p(u^1)$ is determined from $(13)_3$, and $(13)_{1,2}$ is reduced to the equilibrium subsystem (10). Characteristic speeds of the differential part of (13) are

$$\Lambda_1(\mathbf{U}) = -\sqrt{\nu}, \quad \Lambda_2(\mathbf{U}) = 0, \quad \Lambda_3(\mathbf{U}) = \sqrt{\nu}, \tag{14}$$

for $\mathbf{U} = (u^1, u^2, v)^T$. Since $\mathbf{U}_{\mathrm{E}} = (u^1, u^2, p(u^1))^T$, the subcharacteristic condition reads

$$\Lambda_1(\mathbf{U}_{\mathrm{E}}) \le \lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) \le \Lambda_3(\mathbf{U}_{\mathrm{E}})$$

which is satisfied for $p'(u^1) \le \nu$. In the sequel we shall assume that strict inequality holds. Otherwise, a continuous shock structure determined by (13) will seize to exist.
Our aim is to seek for continuous travelling wave solution of the system (13) which connects two equilibrium states – one in front of the wave $\mathbf{U}_{\mathrm{E}0} = (u_0^1, u_0^2, p(u_0^1))$ reached when $x \to +\infty$, with another behind it $\mathbf{U}_{\mathrm{E}1} = (u_1^1, u_1^2, p(u_1^1))$ reached for $x \to -\infty$. We shall therefore assume the solution in the form $\mathbf{U} = \hat{\mathbf{U}}(\xi)$ for $\xi = (x - st)/\varepsilon$ which transforms (13) into a system of ODE's

$$\begin{aligned} s(u^1 - u_1^1) + (u^2 - u_1^2) &= 0, \\ s(u^2 - u_1^2) + (v - p(u_1^1)) &= 0, \\ s\dot{v} + \nu \dot{u}^2 &= v - p(u^1), \end{aligned} \tag{15}$$

where $(\,)^{\cdot} = d(\,)/d\xi$. First two equations are integrated and downstream equilibrium state $v(-\infty) = v_{\mathrm{E}}(-\infty) = p(u_1^1)$ for v is used in the course of integration. Note that stationary points of (15), for which $\dot{\mathbf{U}} = \mathbf{0}$, lie on equilibrium manifold. For equilibrium behind the

wave ($v = p(u_1^1)$) the system is trivially satisfied, whereas for the one in front of the wave ($v = p(u_0^1)$) Rankine-Hugoniot equations (12) are recovered from $(15)_{1,2}$.
By eliminating u^2 and v from $(15)_{1,2}$ the system is reduced to a single ODE which reads

$$\dot{u}^1 = \theta(u^1, s) = \frac{s^2\left(u^1 - u_1^1\right) - p(u^1) + p(u_1^1)}{s\left(s^2 - \nu\right)}. \tag{16}$$

Two remarks are in order for (16):

1. $\theta(u_{0,1}^1, s) = 0$, i.e. stationary points lie on equilibrium manifolds related by the shock wave with speed s through Rankine-Hugoniot equations (12);
2. $\theta(u^1, s)$ has singularity for $s = \Lambda_i(\mathbf{U})$, $i = 1, 2, 3$, i.e. when shock speed coincides with characteristic speeds of the hyperbolic system (13).

It has to be noted that critical value of shock speed $s^* = \lambda_2(\mathbf{u}_1)$, which determines the bifurcation point of the equilibrium system (10), does not have anything in common with characteristic speeds (14) of the hyperbolic system (13). This inevitably motivates stability and bifurcation analysis of the stationary points of (16).
Stability of downstream equilibrium point u_1^1 is determined by the linear variational equation

$$\dot{y} = \Theta(u_1^1, s)y, \quad \Theta(u_1^1, s) = \frac{\partial\theta}{\partial u^1}(u_1^1, s) = \frac{s^2 - p'(u_1^1)}{s\left(s^2 - \nu\right)},$$

where $y = u^1 - u_1^1$ is a perturbation. Eigenvalue is equal to the coefficient $\Theta(u_1^1, s)$ and its sign depends upon the value of shock speed. The critical values of shock speed s occur for

$$\Theta(u_1^1, s_{\mathrm{h}}^*) = 0 \qquad \Rightarrow \qquad s_{\mathrm{h}}^* = \pm\sqrt{p'(u_1^1)},$$

and obviously coincide with characteristic speeds (11) of equilibrium system (10). Taking $s_{\mathrm{h}}^* = s^* = \lambda_2(u_1^1)$ it is easy to check that stability of stationary point is changed in the neighborhood of this critical value since

$$\frac{\partial\Theta}{\partial s}(u_1^1, s_{\mathrm{h}}^*) = -\frac{2}{\nu - p'(u_1^1)} \neq 0. \tag{17}$$

Since $\nu > p'(u_1^1)$, it can be concluded that stability of stationary point u_1^1 is changed in the neighborhood of the critical value of shock speed as follows:
(a) u_1^1 is unstable for $s < s_{\mathrm{h}}^* = \lambda_2(u_1^1)$;
(b) u_1^1 is stable for $s > s_{\mathrm{h}}^* = \lambda_2(u_1^1)$.
In other words, downstream equilibrium state of dissipative hyperbolic system (13) is unstable when shock speed satisfies Lax condition for equilibrium system (10), whereas it is stable when shock speed violates it.
By Taylor expansion of the right hand side of (16) up to second order terms in both variables in the neighborhood of $(u_1^1, s_{\mathrm{h}}^*)$, a bifurcation equation is revealed

$$\dot{y} \approx \frac{1}{2(\nu - p'(u_1^1))}\left(-2\mu y + \frac{p''(u_1^1)}{\sqrt{p'(u_1^1)}}y^2\right), \tag{18}$$

for a bifurcation parameter $\mu = s - s_{\mathrm{h}}^{*}$. This equation describes *transcritical bifurcation* pattern in the neighborhood of the critical value of the parameter $\mu^{*} = 0$.
Although Suliciu's hyperbolic model (13) is physically well founded, it is common in engineering practice to use parabolic models. They are reliable when the state of the system is not far from equilibrium one and nonequilibrium variables are related to equilibrium ones through nonlocal constitutive relations. In this account we shall not pursue in this direction. Interested reader could find in (Simić, 2008) that parabolic model which corresponds to (13) gives rise to the same conclusions about stability and bifurcation as hyperbolic one.
The most important conclusion, which could be drawn from the shock structure analysis in the hyperbolic model of isothermal viscoelasticity (13), is that appearance of the travelling shock profile is indicated by the change of stability of stationary points, i.e. equilibrium states. However, the critical value of the shock speed coincides with its bifurcation value for equilibrium subsystem (10) obtained from Rankine-Hugoniot equations (12). At first, this result is not surprising since stationary points determine the states which lie on equilibrium manifolds $v_{\mathrm{E}} = p(u^{1})$, determined by the source term in $(13)_3$. Nevertheless, the critical shock speed cannot be predicted from the structure of (13) since its characteristic speeds do not coincide with the ones of equilibrium subsystem. This fact emphasizes the role of equilibrium subsystem in the study of shock structure.
A complaint could be made about conclusions since they are based upon reduced system (16) instead of the full system (15) (actually, the system in which $(15)_{1,2}$ are differentiated). Namely, complete downstream equilibrium state $\mathbf{U}_1 = (u_1^1, u_1^2, p(u_1^1))^T$ is non-hyperbolic stationary point of this system for a travelling profile $\mathbf{U}(\xi)$: linearized variational equations have two zero eigenvalues. However, the remaining one has the same behaviour as predicted by linearization of (16) and the conclusion can be retained, albeit in a weaker sense in view of stability.

4. Dissipative hyperbolic model of gas dynamics

Governing equations of gas dynamics are conservation laws of mass, momentum and energy, which in one dimensional setting read

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho v) = 0, \\ &\partial_t(\rho v) + \partial_x(\rho v^2 - \sigma + p) = 0, \\ &\partial_t\left(\frac{1}{2}\rho v^2 + \rho e\right) + \partial_x\left(\left(\frac{1}{2}\rho v^2 + \rho e\right) v - \sigma v + pv + q\right) = 0. \end{aligned} \tag{19}$$

The simplest case is the one in which stress σ and heat flux q are neglected ($\sigma = 0, q = 0$) and one obtains Euler equations of gas dynamics. The vector of state variables $\mathbf{u} = (\rho, v, T)^T$ is consisted of density, velocity and temperature of the gas. Pressure p and internal energy density e are determined by the constitutive equations (thermal and caloric equation of state)

$$p = p(\rho, T) = \rho \frac{k_{\mathrm{B}}}{m} T, \qquad e = e(\rho, T) = \frac{k_{\mathrm{B}}}{m} \frac{T}{\gamma - 1}, \tag{20}$$

where k_{B} is Boltzmann constant, m atomic mass of gas and γ the ratio of specific heats.
In classical approach dissipative effects are modelled through nonlocal constitutive equations for stress and heat flux

$$\sigma = \frac{4}{3}\mu \partial_x v, \qquad q = -\kappa \partial_x T. \tag{21}$$

μ and κ being viscosity and heat conductivity. Actually, (21) describe transport (or nonconvective flux) of momentum and internal energy. By inserting (21) into (19) we obtain so-called Navier-Stokes-Fourier (NSF) model of gas dynamics

$$\begin{aligned}
&\partial_t \rho + \partial_x(\rho v) = 0,\\
&\partial_t(\rho v) + \partial_x(\rho v^2 + p) = \partial_x\left(\frac{4}{3}\mu\partial_x v\right),\\
&\partial_t\left(\frac{1}{2}\rho v^2 + \rho e\right) + \partial_x\left(\left(\frac{1}{2}\rho v^2 + \rho e\right)v + pv\right)\\
&\qquad = \partial_x\left(\kappa\partial_x T + \frac{4}{3}\mu v\partial_x v\right).
\end{aligned} \tag{22}$$

Although it is commonly accepted in engineering applications, it has an intrinsic physical shortcoming: it is parabolic and predicts paradoxical infinite speed of propagation of heat pulses and shear waves.

Another way to describe dissipation is to extend the system of state variables with nonequilibrium ones, i.e. with stress and heat flux $\mathbf{U} = (\rho, v, T, \sigma, q)^T$, and to write appropriate rate-type equations (balance laws) for them. That was systematically analyzed in extended thermodynamics (Müller & Ruggeri, 1998) where the following set of equations is obtained

$$\begin{aligned}
&\partial_t \rho + \partial_x(\rho v) = 0,\\
&\partial_t(\rho v) + \partial_x(\rho v^2 - \sigma + p) = 0,\\
&\partial_t\left(\frac{1}{2}\rho v^2 + \rho e\right) + \partial_x\left(\left(\frac{1}{2}\rho v^2 + \rho e\right)v - \sigma v + pv + q\right) = 0,\\
&\partial_t\left(\rho v^2 + p - \sigma\right) + \partial_x\left(\rho v^3 + 3pv - 3\sigma v + \frac{6}{5}q\right) = \frac{1}{\tau_\sigma}\sigma,\\
&\partial_t\left(\frac{1}{2}\rho v^3 + \frac{5}{2}pv - \sigma v + q\right)\\
&\qquad + \partial_x\left(\frac{1}{2}\rho v^4 + 4pv^2 - \frac{5}{2}\sigma v^2 + \frac{16}{5}qv - \frac{7}{2}\frac{p}{\rho}\sigma + \frac{5}{2}\frac{p^2}{\rho}\right)\\
&\qquad = -\frac{1}{\tau_q}\left(q - \frac{3}{2}\sigma v\right).
\end{aligned} \tag{23}$$

This set of equations is known as *13 moments model*. Closure problem for this model is solved through compatibility with entropy inequality. It has to be noted that the same model was derived in kinetic theory of gases where the balance laws for stress tensor and heat flux came out as macroscopic equations for higher order moments of velocity distribution function (Grad, 1949). That is where the name of the model actually came from.

Small parameters τ_σ and τ_q play the role of relaxation times and, for monatomic gases, obey the relation $\tau_q = 3\tau_\sigma/2$. Although NSF and 13 moments model are principally different, they are related to each other through Chapman-Enskog expansion. Moreover, constitutive equations (21) can be recovered from balance laws of (23) by means of Maxwellian iteration procedure (see Müller & Ruggeri (1998)). Either of these methods lead to the following relations

$$\tau_\sigma = \frac{\mu}{p}, \qquad \tau_q = \frac{2}{5}\frac{\kappa}{p^2}\rho T, \qquad \kappa = \frac{15}{4}\frac{k_B}{m}\mu, \tag{24}$$

valid for monatomic gases. Note that $\mu, \kappa \to 0$, or equivalently $\tau_\sigma, \tau_q \to 0$, pushes the system towards equilibrium manifold $\sigma_{\mathrm{E}} = 0$, $q_{\mathrm{E}} = 0$, i.e. $\mathbf{U}_{\mathrm{E}} = (\rho, v, T, 0, 0)^T$, and leads to the equilibrium subsystem in the form of Euler equations (19) (with $\sigma = 0$, $q = 0$).
It is well-known that characteristic speeds of the equilibrium subsystem (19) correspond to the speed of sound

$$\lambda_1 = v - c_{\mathrm{s}}, \qquad \lambda_2 = v, \qquad \lambda_3 = v + c_{\mathrm{s}}, \tag{25}$$
$$c_{\mathrm{s}} = \left(\gamma \frac{k_B}{m} T\right)^{1/2},$$

whereas characteristic speeds of 13 moments model (23) evaluated on equilibrium manifold for monatomic gases ($\gamma = 5/3$) read

$$\begin{aligned} &\Lambda_1 = v - 1.6503 c_{\mathrm{s}} \qquad \Lambda_2 = v - 0.6297 c_{\mathrm{s}} \\ &\Lambda_3 = v, \\ &\Lambda_4 = v + 0.6297 c_{\mathrm{s}}, \qquad \Lambda_5 = v + 1.6503 c_{\mathrm{s}}. \end{aligned} \tag{26}$$

It is obvious that characteristic speeds of equilibrium subsystem are bounded by the characteristic speeds of 13 moments model

$$\Lambda_1(\mathbf{U}_{\mathrm{E}}) < \lambda_1(\mathbf{u}) < \lambda_3(\mathbf{u}) < \Lambda_5(\mathbf{U}_{\mathrm{E}}),$$

but are not contained in the corresponding spectrum. The aim of this part of the study is to show that travelling shock profile in 13 moments model appears when the shock speed exceeds the highest characteristic speed of the equilibrium subsystem and that it is indicated by the change of stability properties of equilibrium states. All details of the analysis are given in (Simić, 2009); only the main ideas and results will be given here.

4.1 Shock waves in Euler equations

The central point in this stability analysis is subtle relation between complete hyperbolic dissipative system of balance laws and its equilibrium counterpart. Therefore, in the first step it is needed to determine possible equilibrium states related by Rankine-Hugoniot equations for the equilibrium subsystem. Using standard notation and introducing relative velocity of the fluid with respect to shock wave $u = v - s$, the following Rankine-Hugoniot equations are obtained for (19)

$$\begin{aligned} &[[\rho u]] = 0, \\ &\left[\left[\rho u^2 + p\right]\right] = 0, \\ &\left[\left[\left(\frac{1}{2}\rho u^2 + \rho\varepsilon\right) u + pu\right]\right] = 0, \end{aligned}$$

whose nontrivial solution is

$$\begin{aligned} &\frac{\rho_1}{\rho_0} = \frac{M_0^2}{1 - \tilde{\mu}^2(1 - M_0^2)}, \\ &\frac{u_1}{u_0} = \frac{1}{M_0^2}\left[1 - \tilde{\mu}^2(1 - M_0^2)\right], \\ &\frac{T_1}{T_0} = \frac{1}{M_0^2}\left[(1 - \tilde{\mu}^2(1 - M_0^2))((1 + \tilde{\mu}^2)M_0^2 - \tilde{\mu}^2)\right], \end{aligned} \tag{27}$$

where $M_0 = u_0/c_0$ is Mach number in the upstream equilibrium, c_0 is corresponding sound speed and $\tilde{\mu}^2 = (\gamma-1)/(\gamma+1)$. Note that downstream equilibrium state $\mathbf{u}_1$ tends to upstream one $\mathbf{u}_0$ when $M_0 \to 1$, i.e. when the shock speed tends to the highest characteristic speed. In such a way downstream equilibrium can be regarded as bifurcation solution $\mathbf{u}_1(\mathbf{u}_0, M_0)$, and it is transverse to trivial one in the neighborhood of the critical point $M_0 = 1$. For future reference it is important to note that according to Lax condition only the supersonic nontrivial branch ($M_0 > 1$) of the solution (27) is physically admissible, whereas the trivial branch ($\mathbf{u}_1 = \mathbf{u}_0$) is physically admissible in subsonic case ($M_0 < 1$).

4.2 Shock structure in 13 moments model: stability and bifurcation of equilibria

Analysis of the shock structure in 13 moments model (23) will be performed in dimensionless form. Also, it will be assumed that viscosity of the gas is temperature dependent, $\mu = \mu_0(T/T_0)^\alpha$, where exponent α depends on the type of gas and μ_0 is viscosity in reference state. The problem will be considered in a moving reference frame with a single independent variable $\xi = x - st$, and the following dimensionless variables will be used

$$\tilde{\xi} = \frac{\xi}{l_0}, \qquad \tilde{\rho} = \frac{\rho}{\rho_0}, \qquad \tilde{u} = \frac{v-s}{c_0}, \qquad \tilde{T} = \frac{T}{T_0}, \qquad \tilde{\sigma} = \frac{\sigma}{\rho_0 \frac{k_B}{m} T_0}, \qquad \tilde{q} = \frac{q}{\rho_0 \frac{k_B}{m} T_0 c_0},$$

$$M_0 = \frac{v_0 - s}{c_0}, \qquad c_0 = \left(\frac{5}{3}\frac{k_B}{m}T_0\right)^{1/2}, \qquad l_0 = \frac{\mu_0}{\rho_0 \frac{k_B}{m} T_0} c_0.$$

The system of shock structure equations in dimensionless form derived from (23) reads (tildes are dropped for convenience)

$$\begin{aligned}
&\frac{d}{d\xi}(\rho u) = 0\\
&\frac{d}{d\xi}\left(\frac{5}{3}\rho u^2 + \rho T - \sigma\right) = 0,\\
&\frac{d}{d\xi}\left(\frac{5}{6}\rho u^3 + \frac{5}{2}\rho T u - \sigma u + q\right) = 0,\\
&\frac{d}{d\xi}\left(\frac{5}{3}\rho u^3 + 3\rho T u - 3\sigma u + \frac{6}{5}q\right) = \rho T^{1-\alpha}\sigma,\\
&\frac{d}{d\xi}\left(\frac{5}{6}\rho u^4 + 4\rho T u^2 - \frac{5}{2}\sigma u^2 + \frac{16}{5}q u - \frac{21}{10}T\sigma + \frac{3}{2}\rho T^2\right)\\
&\qquad = -\frac{2}{3}\rho T^{1-\alpha}\left(q - \frac{3}{2}\sigma u\right).
\end{aligned} \tag{28}$$

Equations $(28)_{1-3}$ can be integrated taking into account equilibrium values for $\xi \to -\infty$

$$\rho_0 = 1, \qquad u_0 = M_0, \qquad T_0 = 1, \qquad \sigma_0 = 0, \qquad q_0 = 0,$$

and used to express ρ, σ and q in terms of u, T and M_0 as a parameter

$$\begin{aligned}
\rho &= \frac{M_0}{u},\\
\sigma &= \frac{5}{3}M_0(u - M_0) + M_0\frac{T}{u} - 1,\\
q &= \frac{5}{6}M_0\left(M_0^2 - u^2\right) + \frac{5}{2}M_0(1-T) + \frac{5}{3}M_0 u(u - M_0) + M_0 T - u.
\end{aligned}$$

Inserting these relations into $(28)_{4,5}$ one obtains the system of two first-order ODE's

$$\frac{du}{d\xi} = F(u, T, M_0), \qquad \frac{dT}{d\xi} = G(u, T, M_0), \tag{29}$$

whose structure will be omitted for the sake of brevity, but could be found in (Simić, 2009). Although the analytical form of (29) is quite cumbersome, it can be shown that (dimensionless) upstream equilibrium $(u_0, T_0) = (M_0, 1)$ and downstream equilibrium

$$u_1 = \frac{3 + M_0^2}{4M_0}, \qquad T_1 = \frac{1}{16}\left(14 - \frac{3}{M_0^2} + 5M_0^2\right), \tag{30}$$

determined from (27) for $\gamma = 5/3$, are the stationary points, i.e.

$$F(u_0, T_0, M_0) = F(u_1, T_1, M_0) = 0,$$
$$G(u_0, T_0, M_0) = G(u_1, T_1, M_0) = 0.$$

The main idea of stability analysis is to show that stationary points (u_0, T_0) and (u_1, T_1) change their stability properties in the neighborhood of the critical value of upstream Mach number, $M_0 = 1$. In the sense of dynamical system theory they are hyperbolic for $M_0 \neq 1$ but loose hyperbolicity for $M_0 = 1$. This can be proved by analyzing eigenvalues of the linearized system derived from (29) at stationary points. In particular, stationary points are distinct and corresponding linearized systems have nonzero eigenvalues for $M_0 \neq 1$, whereas they coincide for $M_0 = 1$ and one of the eigenvalues is equal zero.
Let $\Delta\mathbf{u} = (\Delta u, \Delta T)^T = (u - u_0, T - T_0)^T = \mathbf{u} - \mathbf{u}_0$ denote the perturbation of upstream equilibrium. Linearized variational equations in the neighborhood of $\mathbf{u}_0$, which read $d(\Delta\mathbf{u})/d\xi = \mathbf{A}_0(M_0)\Delta\mathbf{u}$, have the Jacobian matrix

$$\mathbf{A}_0(M_0) = \frac{1}{27 - 78M_0^2 + 25M_0^4}\begin{pmatrix} \frac{-9+42M_0^2-25M_0^4}{M_0} & 3(3 + M_0^2) \\ \frac{54-162M_0^2+200M_0^4}{9M_0^2} & \frac{-18+114M_0^2-50M_0^4}{3M_0} \end{pmatrix},$$

and corresponding eigenvalues are

$$\lambda_{01}(M_0) = \frac{5D_{0-}}{6M_0(27 - 78M_0^2 + 25M_0^4)}, \qquad \lambda_{02}(M_0) = \frac{5D_{0+}}{6M_0(27 - 78M_0^2 + 25M_0^4)},$$
$$D_{0\mp} = -9 + 48M_0^2 - 25M_0^4 \mp \sqrt{81 - 216M_0^2 + 234M_0^4 + 72M_0^6 + 25M_0^8}.$$

Simple calculation shows

$$\lambda_{01}(1) = 0, \qquad \frac{d\lambda_{01}(1)}{dM_0} = \frac{10}{7}; \qquad \lambda_{02}(1) = -\frac{35}{39}. \tag{31}$$

By continuity argument it follows that there is a neighborhood of $M_0 = 1$ in which $\lambda_{01}(M_0) < 0$ for $M_0 < 1$ and $\lambda_{01}(M_0) > 0$ for $M_0 > 1$, while $\lambda_{02}(M_0) < 0$ in either case. Equivalent arguments, but more cumbersome analytical expressions which will be omitted here, lead to the same results in downstream equilibrium $\mathbf{u}_1$. Namely,

$$\lambda_{11}(1) = 0, \qquad \frac{d\lambda_{11}(1)}{dM_0} = -\frac{10}{7}; \qquad \lambda_{12}(1) = -\frac{35}{39}, \tag{32}$$

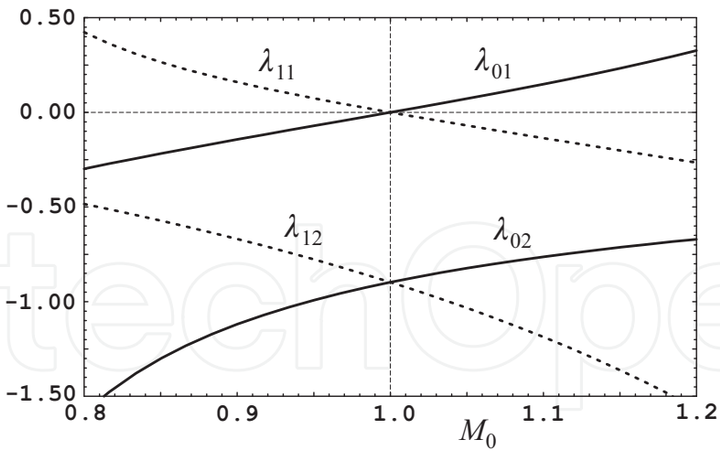

Fig. 1. Eigenvalues in upstream $\mathbf{u}_0$ and downstream $\mathbf{u}_1$ stationary points in gas dynamics shock structure.

which by continuity argument lead to a conclusion that there is a neighborhood of $M_0 = 1$ in which $\lambda_{11}(M_0) > 0$ for $M_0 < 1$ and $\lambda_{11}(M_0) < 0$ for $M_0 > 1$, while $\lambda_{12}(M_0) < 0$ in either case. In the language of dynamical system theory, $\mathbf{u}_0$ is a stable node and $\mathbf{u}_1$ is a saddle when $M_0 < 1$, while $\mathbf{u}_0$ is a saddle and $\mathbf{u}_1$ is a stable node when $M_0 > 1$. Obviously, this characterization is valid only for the reduced system of shock structure equations (29): in the complete system (28) neither of stationary points which lie on equilibrium manifold is hyperbolic, but there is always one eigenvalue which changes the sign in the neighborhood of $M_0 = 1$ and indicates the critical value of the shock speed. This property has already been observed in the hyperbolic model of isothermal viscoelasticity.

Like in the previous section, bifurcation analysis should reveal the type of bifurcation which occur in the neighborhood of the critical value of shock speed. Unlike the shock structure problem (16) which was one-dimensional, shock structure equations (29) in 13 moments model form a two-dimensional system. In such a case recognition of bifurcation pattern have to be preceded by the reduction process. In this problem the centre manifold reduction could be efficiently applied (see Guckenheimer & Holmes (1986) for a detailed account on this method). Complete procedure is given in (Simić, 2009); only essential steps will be shown in this review. First, the system (29) is extended by trivial equation $dM_0/d\xi = 0$ and analyzed in the neighborhood of the stationary point $\mathbf{u}_0$. Let $\Delta M_0 = M_0 - 1$. By means of linear transformation

$$(\Delta u, \Delta T, \Delta M_0) = (-3y/2 + 18z/23, y + z, \epsilon)$$

the extended system is transformed into normal form

$$\frac{d}{d\xi}\begin{pmatrix} y \\ z \\ \epsilon \end{pmatrix} = \begin{pmatrix} f(y,z,\epsilon) \\ g(y,z,\epsilon) \\ 0 \end{pmatrix} = \mathbf{B}_0(\epsilon)\begin{pmatrix} y \\ z \\ \epsilon \end{pmatrix} + \begin{pmatrix} Y(y,z,\epsilon) \\ Z(y,z,\epsilon) \\ 0 \end{pmatrix}, \tag{33}$$

where

$$\mathbf{B}_0(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -35/39 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{34}$$

and $Y(y,z,\epsilon)$ and $Z(y,z,\epsilon)$ are at least of the second order in y and z. Centre manifold is locally invariant manifold in the neighborhood of the critical point; it is tangent to the eigenspace

spanned by the eigenvectors corresponding to eigenvalues with zero real part. Due to the structure of (34) it has the form $z = h(y, \epsilon)$ and should satisfy conditions

$$h(0,0) = 0, \qquad \frac{\partial h(0,0)}{\partial y} = 0, \qquad \frac{\partial h(0,0)}{\partial \epsilon} = 0.$$

Determining equation for the centre manifold comes from equation for z variable in (33) and reads

$$\frac{\partial h(y,\epsilon)}{\partial y} f(y, h(y,\epsilon), \epsilon) - g(y, h(y,\epsilon), \epsilon) = 0.$$

Since it cannot be resolved analytically, the approximate solution can be found using power series expansion

$$h(y,\epsilon) \approx \frac{184}{245}\epsilon y - \frac{115}{196} y^2.$$

Insertion of this result into (33) yields the bifurcation equation

$$\frac{dy}{d\xi} \approx \frac{10}{7}\epsilon y - \frac{20}{7} y^2 \tag{35}$$

which represents the canonical form of transcritical bifurcation pattern. It is well known that (trivial) stationary point $y_0 = 0$ looses its stability for $\epsilon > 0$, while nontrivial one $y_1 \approx \epsilon/2$ becomes stable. This exchange of stability of stationary points described by the lumped variable y reflects the change of stability properties of the equilibrium states $\mathbf{u}_0$ and $\mathbf{u}_1$.

Stability and bifurcation analysis of equilibrium states in 13 moments model, related to the shock structure problem, has dominantly mathematical character. The main question is: can we draw a conclusion of thermodynamical flavor from these results? The idea will come from comparison of stability results (31)-(32) with the Lax condition. By the Lax condition, shock waves in Euler model, i.e. nontrivial bifurcating solution of Rankine-Hugoniot equations, are admissible when shock speed s satisfies the inequality

$$\lambda_3(\mathbf{u}_0) < s < \lambda_3(\mathbf{u}_1).$$

Left inequality corresponds to $M_0 > 1$. When $s < \lambda_3(\mathbf{u}_0)$, or equivalently $M_0 < 1$, shocks are not admissible. These conditions are in clear correspondence with stability results for 13 moments model. Eigenvalue $\lambda_{01}(M_0) < 0$ and determines the stable direction (eigenvector) in state space when $M_0 < 1$. On the other hand, $\lambda_{01}(M_0) > 0$ and determines the unstable direction in state space for $M_0 > 1$. Recall that another eigenvalue is not of any importance since it does not change the sign. The opposite statements are valid for the eigenvalue $\lambda_{11}(M_0)$. Therefore, we can conclude that admissible shock structure appears when

$$0 < \lambda_{01}(M_0) \qquad (\lambda_{11}(M_0) < 0), \tag{36}$$

while $\lambda_{01}(M_0) = 0$, or $\lambda_{11}(M_0) = 0$, determine the critical value of the parameter (shock speed). These conditions seem to be the natural consequence of the fact that Euler system (19) is contained in 13 moments model (23) and the properties of equilibrium system are "absorbed" by the dissipative one. As typical for selection rules, they are expressed in the form of inequality and reflect irreversible character of the process.

Although it looks like a simple and natural transfer of analysis from equilibrium to a dissipative system, the main point is that equilibrium states (stationary points), the critical value of the shock speed and stability condition (36) could be determined from the shock

structure equations *without prior knowledge* of solutions of Rankine-Hugoniot equations and exploitation of Lax condition. This self-contained analysis of the shock structure in hyperbolic dissipative systems will be the cornerstone in studying the same problem in binary mixture. It was shown in (Simić, 2009) that completely the same conclusions can be drawn even for the hyperbolic models with higher order moments (14 and 21), as well as for parabolic NSF model. This suggests that stability properties and bifurcation pattern are not the same by mere coincidence, but reflect intrinsic properties of dissipative systems.

5. Multi-temperature mixture of Euler fluids

The mixture model we shall deal with in this account arose within the framework of rational thermodynamics (Truesdell, 1969). Roughly speaking, it is based upon assumption that behavior of each fluid component is governed by the same balance laws as a single fluid, but mutual interactions among them are taken into account through appropriate source terms. On the other hand, governing equations for the mixture ought to be reconstructed from equations for components. However, these general principles allow for different levels of accuracy depending on the number of variables chosen to describe the state of each component.
Since the modelling issue is very important for the study of shock structure, the first part of this section will be devoted to the review of recent results about macroscopic multi-temperature model. The general model will then be specialized to the case of binary mixture for which the shock structure will be analyzed in the same spirit as it was done in the previous section.

5.1 The model and its properties

According to assumptions given above, the model of the mixture is consisted of balance laws of mass, momentum and energy for each component $\alpha = 1, \ldots, n$

$$\begin{aligned}
&\frac{\partial \rho_\alpha}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha) = \tau_\alpha, \\
&\frac{\partial}{\partial t}(\rho_\alpha \mathbf{v}_\alpha) + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha - \mathbf{t}_\alpha) = \mathbf{m}_\alpha, \\
&\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_\alpha v_\alpha^2 + \rho_\alpha \varepsilon_\alpha\right) + \operatorname{div}\left\{\left(\frac{1}{2}\rho_\alpha v_\alpha^2 + \rho_\alpha \varepsilon_\alpha\right)\mathbf{v}_\alpha - \mathbf{t}_\alpha \mathbf{v}_\alpha + \mathbf{q}_\alpha\right\} = e_\alpha.
\end{aligned} \tag{37}$$

By ρ_α, $\mathbf{v}_\alpha$, ε_α, $\mathbf{t}_\alpha$ and $\mathbf{q}_\alpha$ we denote respectively the mass density, the velocity, the internal energy density, the stress tensor and the heat flux of each component. Source terms τ_α, $\mathbf{m}_\alpha$ and e_α describe the actions of other components which cause the change of mass, momentum and energy density of the chosen one. In the sequel it will be assumed that all the components of the mixture are Euler fluids, i.e. they are neither viscous, nor heat-conducting, so that stress tensor and heat flux are assumed as

$$\mathbf{t}_\alpha = -p_\alpha \mathbf{I}, \qquad \mathbf{q}_\alpha = \mathbf{0}, \tag{38}$$

where p_α is the partial pressure. In such a model, the state of each component is determined by its own mass density, velocity and temperature field, $(\rho_\alpha, \mathbf{v}_\alpha, T_\alpha)^T$, thus the name *multi-temperature* mixture. This approach is not a widespread one in continuum modelling of mixtures, except in the case of plasma. However, it naturally arises in macroscopic equations derived from kinetic theory of gases, i.e. Boltzmann equations for mixtures, either due to presence of distribution functions for each specie (Bisi et al., 2005), or due to violation

of equipartition of energy for translational degrees of freedom (Hoover & Hoover, 2009; Xu & Liu, 2007). In this study the temperatures of components were introduced with the aim to show that, in the absence of viscosity and heat conductivity, large mass disparity is the main driving agent for thermal nonequilibrium.
To reconstruct the governing equations for the mixture we need some more assumptions and definitions. It will be assumed that source terms satisfy the following relations

$$\sum_{\alpha=1}^{n} \tau_\alpha = 0, \qquad \sum_{\alpha=1}^{n} \mathbf{m}_\alpha = \mathbf{0}, \qquad \sum_{\alpha=1}^{n} e_\alpha = 0. \tag{39}$$

Along with them the following definitions for field variables of the whole mixture will be introduced

$$\begin{aligned}
&\rho = \sum_{\alpha=1}^{n} \rho_\alpha, && \text{mass density,}\\
&\mathbf{v} = \frac{1}{\rho} \sum_{\alpha=1}^{n} \rho_\alpha \mathbf{v}_\alpha, && \text{velocity,}\\
&\mathbf{u}_\alpha = \mathbf{v}_\alpha - \mathbf{v}, && \text{diffusion velocities,}\\
&\varepsilon_I = \frac{1}{\rho} \sum_{\alpha=1}^{n} \rho_\alpha \varepsilon_\alpha, && \text{intrinsic internal energy density,}\\
&\varepsilon = \varepsilon_I + \frac{1}{2\rho} \sum_{\alpha=1}^{n} \rho_\alpha u_\alpha^2, && \text{internal energy density}\\
&\mathbf{t} = \sum_{\alpha=1}^{n} \left(\mathbf{t}_\alpha - \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha \right), && \text{stress tensor,}\\
&\mathbf{q} = \sum_{\alpha=1}^{n} \left\{ \mathbf{q}_\alpha + \rho_\alpha \left(\varepsilon_\alpha + \frac{1}{2} u_\alpha^2 \right) \mathbf{u}_\alpha - \mathbf{t}_\alpha \mathbf{u}_\alpha \right\}, && \text{flux of internal energy (heat flux).}
\end{aligned} \tag{40}$$

Summing up equations (37) one obtains the conservation laws of mass, momentum and energy for the whole mixture

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0,\\
&\frac{\partial}{\partial t}(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} - \mathbf{t}) = \mathbf{0},\\
&\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + \rho \varepsilon \right) + \operatorname{div} \left\{ \left(\frac{1}{2} \rho v^2 + \rho \varepsilon \right) \mathbf{v} - \mathbf{t}\mathbf{v} + \mathbf{q} \right\} = 0.
\end{aligned} \tag{41}$$

Note that assumptions (38) lead to a simplified form of the stress tensor $\mathbf{t}$ and the heat flux $\mathbf{q}$. However, they are not simple componentwise sums of corresponding quantities, but are influenced by the diffusion process through $\mathbf{u}_\alpha$

$$\mathbf{t} = -p\mathbf{I} - \sum_{\alpha=1}^{n} \left(\rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha \right); \qquad p = \sum_{\alpha=1}^{n} p_\alpha, \qquad \mathbf{q} = \sum_{\alpha=1}^{n} \left\{ \rho_\alpha \left(\varepsilon_\alpha + \frac{1}{2} u_\alpha^2 \right) + p_\alpha \right\} \mathbf{u}_\alpha,$$

where p denotes the total pressure of the mixture. Our concern will be even restricted class of Euler fluids – perfect gases whose constitutive equations are

$$p_\alpha = \rho_\alpha \frac{k_\mathrm{B}}{m_\alpha} T_\alpha, \qquad \varepsilon_\alpha = \frac{k_\mathrm{B}}{m_\alpha} \frac{T_\alpha}{\gamma_\alpha - 1}. \tag{42}$$

This completes the results based upon principles of rational thermodynamics.

The structure of source terms in (37) is determined using the general principles of extended thermodynamics – Galilean invariance and the entropy principle. Galilean invariance (Ruggeri, 1989) restricts the velocity dependence of source terms to the following form (Ruggeri & Simić, 2007):

$$\tau_b = \hat{\tau}_b, \qquad \mathbf{m}_b = \hat{\tau}_b \mathbf{v} + \hat{\mathbf{m}}_b, \qquad e_b = \hat{\tau}_b \frac{v^2}{2} + \hat{\mathbf{m}}_b \cdot \mathbf{v} + \hat{e}_b, \tag{43}$$

for $b = 1, \ldots, n-1$, where $\hat{\tau}_b$, $\hat{\mathbf{m}}_b$ and $\hat{e}_b$ are independent of the mixture velocity $\mathbf{v}$. The final form of velocity independent parts of source terms is determined by means of entropy inequality (Ruggeri & Simić, 2007)

$$\begin{aligned}
\hat{\tau}_b &= -\sum_{c=1}^{n-1} \varphi_{bc} \left(\frac{\mu_c - \frac{1}{2}u_c^2}{T_c} - \frac{\mu_n - \frac{1}{2}u_n^2}{T_n} \right), \\
\hat{\mathbf{m}}_b &= -\sum_{c=1}^{n-1} \psi_{bc} \left(\frac{\mathbf{u}_c}{T_c} - \frac{\mathbf{u}_n}{T_n} \right), \\
\hat{e}_b &= -\sum_{c=1}^{n-1} \theta_{bc} \left(-\frac{1}{T_c} + \frac{1}{T_n} \right),
\end{aligned}$$

where:

$$\mu_\alpha = \varepsilon_\alpha - T_\alpha S_\alpha + \frac{p_\alpha}{\rho_\alpha}, \qquad (\alpha = 1, \ldots, n)$$

are chemical potentials of the constituents and φ_{bc}, ψ_{bc} and θ_{bc}, $(b, c = 1, \ldots, n-1)$ are phenomenological symmetric positive definite matrices. Source terms are given for $n-1$ components since the last one can be determined from restrictions (39).

One of the most important properties of the mixture model (37) is its hyperbolicity. Characteristic speeds calculated from the differential part are almost trivial – they consist of independent subsets of characteristic speeds of the components

$$\lambda_\alpha^{(1)} = v_{\alpha\mathrm{n}} - c_{\mathrm{s}\alpha}, \qquad \lambda_\alpha^{(2,3,4)} = v_{\alpha\mathrm{n}}, \qquad \lambda_\alpha^{(5)} = v_{\alpha\mathrm{n}} + c_{\mathrm{s}\alpha},$$
$$c_{\mathrm{s}\alpha} = \left(\gamma_\alpha \frac{k_\mathrm{B}}{m_\alpha} T_\alpha \right)^{1/2},$$

where $v_{\alpha\mathrm{n}} = \mathbf{v}_\alpha \cdot \mathbf{n}$ are component velocities normal to the wave front and $c_{\mathrm{s}\alpha}$ are local sound speeds of the components.

It is crucial for further study to use the possibility to substitute the balance laws for one component from (37), say n, with conservation laws for the mixture (41)

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0,\\
&\frac{\partial}{\partial t}(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} - \mathbf{t}) = \mathbf{0},\\
&\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) + \operatorname{div}\left\{\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right)\mathbf{v} - \mathbf{t}\mathbf{v} + \mathbf{q}\right\} = 0,\\
&\frac{\partial \rho_b}{\partial t} + \operatorname{div}(\rho_b \mathbf{v}_b) = \tau_b,\\
&\frac{\partial}{\partial t}(\rho_b \mathbf{v}_b) + \operatorname{div}(\rho_b \mathbf{v}_b \otimes \mathbf{v}_b - \mathbf{t}_b) = \mathbf{m}_b,\\
&\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_b v_b^2 + \rho_b\varepsilon_b\right) + \operatorname{div}\left\{\left(\frac{1}{2}\rho_b v_b^2 + \rho_b\varepsilon_b\right)\mathbf{v}_b - \mathbf{t}_b\mathbf{v}_b + \mathbf{q}_b\right\} = e_b.
\end{aligned} \tag{44}$$

These two sets of governing equations, (37) and (44), are equivalent – they are actually obtained by the change of state variables $(\rho_\alpha, \mathbf{v}_\alpha, T_\alpha)^T \rightarrow (\rho, \mathbf{v}, T, \rho_b, \mathbf{v}_b, T_b)^T$. As a consequence, their characteristic speeds are the same.
In this setting $\mathbf{v}_b$ (or $\mathbf{u}_b$) and T_b can be regarded as nonequilibrium variables. As a matter of fact, mixture theory was, in its historical development, formulated within extended thermodynamics as a single-temperature theory, but with component velocities $\mathbf{v}_b$ as state variables. That was crucial for removal of paradox of infinite speed of pulse propagation due to the presence of nonlocal constitutive equation – Fick law. The hyperbolic single-temperature system appears as a principal subsystem of (44). Complete hierarchy of principal subsystems was analyzed in (Ruggeri & Simić, 2007) and it was shown that in the case of non-reacting mixtures equilibrium condition reads

$$\mathbf{v}_1 = \ldots = \mathbf{v}_n = \mathbf{v}; \qquad (\mathbf{u}_1 = \ldots = \mathbf{u}_n = \mathbf{0}), \qquad T_1 = \ldots = T_n = T, \tag{45}$$

and leads to an equilibrium subsystem (41). Corresponding characteristic speeds are

$$\lambda_{\mathrm{E}}^{(1)} = v_{\mathrm{n}} - c_{\mathrm{sE}}, \qquad \lambda_{\mathrm{E}}^{(2,3,4)} = v_{\mathrm{n}}, \qquad \lambda_{\mathrm{E}}^{(5)} = v_{\mathrm{n}} + c_{\mathrm{sE}},$$

where $v_{\mathrm{n}} = \mathbf{v} \cdot \mathbf{n}$ and c_{sE} is equilibrium sound speed of the mixture. Since (44) is endowed with convex entropy, the following subcharactiristic condition is valid

$$\min_\alpha \left(v_{\alpha \mathrm{n}} - c_{\mathrm{s}\alpha}^*\right) \leq \lambda_{\mathrm{E}}^{(1)}, \qquad \lambda_{\mathrm{E}}^{(5)} \leq \max_\alpha \left(v_{\alpha \mathrm{n}} + c_{\mathrm{s}\alpha}^*\right)$$

where $c_{\mathrm{s}\alpha}^* = (\gamma_\alpha (k_{\mathrm{B}}/m_\alpha) T)^{1/2}$ are the sound speeds of the full system evaluated in equilibrium. Explicit form of the equilibrium sound speed of the mixture c_{sE} will be left for the case of binary mixture. It is only important to note that it is evaluated when (45) is satisfied.

5.2 Binary mixture and the shock structure problem

Analysis of the shock structure in mixtures, modelled by the hyperbolic system (44), will be restricted to a binary mixture of perfect gases without chemical reactions ($\tau_\alpha = 0$). It was

shown (Ruggeri, 2001) that governing equations (44) could be written in the following form in the case of binary mixture

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0,\\
&\frac{\partial}{\partial t}(\rho \mathbf{v}) + \operatorname{div}\left(\rho \mathbf{v} \otimes \mathbf{v} + p\mathbf{I} + \frac{1}{\rho c(1-c)}\mathbf{J} \otimes \mathbf{J}\right) = \mathbf{0},\\
&\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) + \operatorname{div}\left\{\left(\frac{1}{2}\rho v^2 + \rho\varepsilon + p\right)\mathbf{v} + \left(\frac{\mathbf{v}\cdot\mathbf{J}}{\rho c(1-c)} + \frac{1}{\beta}\right)\mathbf{J}\right\} = 0,\\
&\frac{\partial}{\partial t}(\rho c) + \operatorname{div}(\rho c \mathbf{v} + \mathbf{J}) = 0,\\
&\frac{\partial}{\partial t}(\rho c \mathbf{v} + \mathbf{J}) + \operatorname{div}\left\{\rho c \mathbf{v} \otimes \mathbf{v} + \frac{1}{\rho c}\mathbf{J} \otimes \mathbf{J} + \mathbf{v} \otimes \mathbf{J} + \mathbf{J} \otimes \mathbf{v} + \nu \mathbf{I}\right\} = \mathbf{m}_1,\\
&\frac{\partial}{\partial t}\left(\frac{1}{2}\rho c\left(\mathbf{v} + \frac{\mathbf{J}}{\rho c}\right)^2 + \rho c e\right) + \operatorname{div}\left\{\left(\frac{1}{2}\rho c\left(\mathbf{v} + \frac{\mathbf{J}}{\rho c}\right)^2 + \rho c e + \nu\right)\left(\mathbf{v} + \frac{\mathbf{J}}{\rho c}\right)\right\} = e_1,
\end{aligned} \tag{46}$$

provided we introduce concentration variable c and diffusion flux vector $\mathbf{J}$

$$c = \frac{\rho_1}{\rho}, \qquad \mathbf{J} = \rho_1 \mathbf{u}_1 = -\rho_2 \mathbf{u}_2.$$

Thermal inertia β reads

$$\beta = \frac{1}{g_1 - g_2}, \qquad g_\alpha = \varepsilon_\alpha + \frac{p_\alpha}{\rho_\alpha} + \frac{u_\alpha^2}{2},$$

and $\nu = p_1$, $e = \varepsilon_1$.
Thermal and caloric equation of state for the mixture could be expressed in the form of equations for a single fluid

$$p = \rho \frac{k_B}{m} T, \qquad \varepsilon_I = \frac{k_B}{m}\frac{T}{\gamma - 1}, \tag{47}$$

provided we introduce average atomic mass m, average temperature T and average ratio of specific heats γ in the following way (Ruggeri & Simić, 2007)

$$\begin{aligned}
&\frac{1}{m} = \frac{c}{m_1} + \frac{1-c}{m_2}, \qquad T = c\frac{m}{m_1}T_1 + (1-c)\frac{m}{m_2}T_2,\\
&\frac{1}{\gamma - 1} = \frac{c}{\gamma_1 - 1}\frac{m}{m_1}\frac{T_1}{T} + \frac{1-c}{\gamma_2 - 1}\frac{m}{m_2}\frac{T_2}{T}.
\end{aligned} \tag{48}$$

Note that $\gamma_1 = \gamma_2$ implies $\gamma = \gamma_1 = \gamma_2$. It will be useful for further analysis to use the temperature difference $\Theta = T_2 - T_1$ as a nonequilibrium variable, so that component temperatures can expressed in terms of Θ and average temperature T

$$T_1 = T - \frac{m}{m_2}(1-c)\Theta, \qquad T_2 = T + \frac{m}{m_1}c\Theta. \tag{49}$$

Using these relations ν, e and β can be expressed in terms of $\mathbf{u} = (\rho, \mathbf{v}, T, c, \mathbf{J}, \Theta)^T$, a new set of state variables. Using these new quantities one may determine the sound speed for equilibrium subsystem

$$c_{\mathrm{sE}} = \left(\gamma \frac{k_{\mathrm{B}}}{m} T\right)^{1/2}. \tag{50}$$

Explicit form of source terms $\mathbf{m}_1$ and e_1 reads as follows

$$\hat{\mathbf{m}}_1 = -\psi_{11}\left(\frac{\mathbf{u}_1}{T_1} - \frac{\mathbf{u}_2}{T_2}\right) = -\psi_{11}\left(\frac{1}{\rho c T_1} + \frac{1}{\rho(1-c)T_2}\right)\mathbf{J},$$
$$\hat{e}_1 = -\theta_{11}\left(-\frac{1}{T_1} + \frac{1}{T_2}\right) = \theta_{11}\frac{\Theta}{T_1 T_2},$$

where T_1 and T_2 are given by (49). However, source terms cannot be completely described within the framework of extended thermodynamics since coefficients ψ_{11} and θ_{11} remain undetermined. They can be related either to experimental data, or to some other theory. We shall recourse to the results of kinetic theory of gases (Bose, 2004) which relate them to volumetric collision frequency Γ'_{12} in equilibrium

$$\psi_{11\mathrm{E}} = 2T_0\frac{m_1 m_2}{m_1 + m_2}\Gamma'_{12}, \qquad \theta_{11\mathrm{E}} = 3k_{\mathrm{B}}T_0^2\frac{m_1 m_2}{(m_1 + m_2)^2}\Gamma'_{12}.$$

On the other hand, by linearization of source terms in the neighborhood of equilibrium they could be related to relaxation times τ_{D} and τ_{T} for diffusion and temperature in equilibrium

$$\psi_{11\mathrm{E}} = \frac{\rho_0 c_0(1-c_0)T_0}{\tau_{\mathrm{D}}}, \qquad \theta_{11\mathrm{E}} = \frac{k_{\mathrm{B}}}{m_1(\gamma_1 - 1)}\frac{\rho_0 c_0 T_0^2}{\tau_{\mathrm{T}}}. \tag{51}$$

Comparison of the last two equations yields the relation between relaxation times

$$\tau_{\mathrm{T}} = \frac{2}{3(\gamma_1 - 1)}\frac{m_1 + m_2}{(1-c_0)m_1}\tau_{\mathrm{D}}$$

and it is frank that for real gases we have an estimate $\tau_{\mathrm{T}} > \tau_{\mathrm{D}}$. This fact gives strong support to the multi-temperature assumption since thermal nonequilibrium needs more time for attenuation than mechanical (diffusion) one. In the sequel we shall adopt equilibrium values of coefficients (51): since we are dealing with states which are close to equilibrium ones, they will provide sufficiently accurate estimate.

Equilibrium state is characterized by vanishing of the entropy production. In the case of binary mixture it is equivalent to vanishing of the source terms, $\mathbf{m}_1 = \mathbf{0}$ and $e_1 = 0$ (see Ruggeri & Simić (2007)). Condition of equilibrium leads to a state without diffusion where constituents have common temperature

$$\mathbf{J} = \mathbf{J}_0 = \mathbf{0}, \qquad \Theta = \Theta_0 = 0.$$

These conditions are in agreement with (45). We shall further assume that other variables have uniform distributions in equilibrium state

$$\rho = \rho_0 = \text{const.}, \qquad \mathbf{v} = \mathbf{v}_0 = \text{const.}, \qquad T = T_0 = \text{const.}, \qquad c = c_0 = \text{const.}$$

The shock structure is assumed to be a continuous travelling wave-like solution which asymptotically connects two such equilibrium states.

The shock structure problem will be analyzed for a plane wave which propagates in x-direction with speed s. Therefore, we shall seek for a travelling wave solution in the form $\mathbf{u}(\xi) = \mathbf{u}(x - st)$ with a single independent variable $\xi = x - st$. This assumption transforms

the system of governing equations (46) to the following set of ODE's which determine the shock structure

$$\begin{aligned}
&\frac{d}{d\xi}(\rho u) = 0,\\
&\frac{d}{d\xi}\left(\rho u^2 + p + \frac{J^2}{\rho c(1-c)}\right) = 0,\\
&\frac{d}{d\xi}\left\{\left(\frac{1}{2}\rho u^2 + \rho\varepsilon + p\right)u + \left(\frac{uJ}{\rho c(1-c)} + \frac{1}{\beta}\right)J\right\} = 0,\\
&\frac{d}{d\xi}(\rho c u + J) = 0,\\
&\frac{d}{d\xi}\left\{\rho c u^2 + \frac{J^2}{\rho c} + 2uJ + \nu\right\} = \hat{m}_1,\\
&\frac{d}{d\xi}\left\{\left(\frac{1}{2}\rho c\left(u + \frac{J}{\rho c}\right)^2 + \rho c e + \nu\right)\left(u + \frac{J}{\rho c}\right)\right\} = \hat{m}_1 u + \hat{e}_1,
\end{aligned} \tag{52}$$

where $u = v - s$ is x-component of the relative mixture velocity with respect to the shock wave and J and $\hat{m}_1$ are x-components of diffusion flux and source term. Due to change of variables we shall have a standing shock profile with upstream (unperturbed) state $\mathbf{u}_0 = \lim_{\xi\to-\infty}\mathbf{u}(\xi)$ and downstream (perturbed) state $\mathbf{u}_1 = \lim_{\xi\to\infty}\mathbf{u}(\xi)$. In view of equilibrium conditions discussed above, we adjoin the system (52) with following boundary conditions

upstream		downstream	
$\rho = \rho_0,$	$c = c_0;$	$\rho = \rho_1,$	$c = c_1;$
$u = u_0,$	$J = 0;$	$u = u_1,$	$J = 0;$
$T = T_0,$	$\Theta = 0;$	$T = T_1,$	$\Theta = 0.$

(53)

Downstream boundary conditions $\mathbf{u}_1$ will be determined in the sequel.

5.3 Stability and bifurcation of equilibria in shock structure problem

Further analysis will be performed in dimensionless form using the following dimensionless variables

$$\hat{\rho} = \frac{\rho}{\rho_0}, \quad \hat{u} = \frac{u}{c_{\mathrm{sE}}}, \quad \hat{T} = \frac{T}{T_0}, \quad \hat{J} = \frac{J}{\rho_0 c_0 c_{\mathrm{sE}}}, \quad \hat{\Theta} = \frac{\Theta}{T_0}, \quad \hat{\xi} = \frac{\xi}{\tau_{\mathrm{D}} c_{\mathrm{sE}}}, \quad M_0 = \frac{u_0}{c_{\mathrm{sE}}}.$$

We shall also assume that both components have the same ratio of specific heats, equal to the average one γ. Upstream boundary data thus read

$$\hat{\rho}_0 = 1, \quad \hat{u}_0 = M_0, \quad \hat{T}_0 = 1, \quad \hat{J}_0 = 0, \quad \hat{\Theta}_0 = 0, \tag{54}$$

while c_0 is already a dimensionless quantity. For convenience hats will be dropped in the sequel.

To reduce the order of the system, as well as computational effort, conservation laws for mass and momentum of the mixture $(52)_{1,2}$ and conservation law for mass of the component $(52)_4$

will be used to express ρ, J and T in terms of c and u

$$\rho = \frac{M_0}{u}, \qquad J = M_0\left(1 - \frac{c}{c_0}\right), \tag{55}$$
$$T = \frac{m}{m_0}\frac{u}{M_0}\left\{1 + \gamma M_0^2\left[1 - \frac{u}{M_0}\left(1 + \frac{(c_0 - c)^2}{c(1-c)}\right)\right]\right\},$$

where m_0 denotes the average atomic mass evaluated in upstream equilibrium state.
There will remain three equations which will finally determine the shock structure. Although still remains one conservation law among them $(52)_3$ further elimination of state variables will cause appearance of singularities which prevent efficient numerical calculations in the sequel. Thus, inserting (55) into $(52)_{3,5,6}$ written in dimensionless form we formally obtain the following set of ODE's

$$\frac{du}{d\xi} = F(u, c, \Theta, M_0), \qquad \frac{dc}{d\xi} = G(u, c, \Theta, M_0), \qquad \frac{d\Theta}{d\xi} = H(u, c, \Theta, M_0), \tag{56}$$

The actual form of the ODE system is a bit different

$$\mathbf{B}(\mathbf{u}, M_0)\frac{d\mathbf{u}}{d\xi} = \mathbf{f}(\mathbf{u}, M_0) \tag{57}$$

for $\mathbf{u} = (u, c, \Theta)^T$. Under regularity assumption $\det \mathbf{B}(\mathbf{u}, M_0) \neq 0$ equations (56) can be reconstructed as $d\mathbf{u}/d\xi = \mathbf{F}(\mathbf{u}, M_0) = \mathbf{B}^{-1}(\mathbf{u}, M_0)\mathbf{f}(\mathbf{u}, M_0)$. However, the form (57) is much more convenient and will be used in stability analysis.
It is of utmost importance to notice that stationary points of (56), or (57), which satisfy

$$\mathbf{f}(\mathbf{u}_0, M_0) = \mathbf{f}(\mathbf{u}_1, M_0) = \mathbf{0}$$

are determined by relations

$$\mathbf{u}_0 = (u_0, c_0, \Theta_0) = (M_0, c_0, 0), \qquad \mathbf{u}_1 = (u_1, c_1, \Theta_1) = \left(\frac{3 + M_0^2}{4M_0}, c_0, 0\right). \tag{58}$$

Actually, they satisfy the Rankine-Hugoniot equations on singular surface for the equilibrium subsystem (41). This fact is far from being obvious, but it is in a full accordance with the results obtained thus far for other hyperbolic dissipative systems (isothermal viscoelasticity and gas dynamics).
Our main concern will be the stability analysis of stationary points (58). We want show that there is a critical value for the upstream Mach number M_0 for which stationary points change their stability properties. Moreover, we would like to show that we can rely on a simple criterion, related to the eigenvalues, which provides us a selection rule for admissible shock structures. To that end we shall introduce perturbation of upstream equilibrium state $\Delta\mathbf{u} = (u - u_0, c - c_0, \Theta - \Theta_0)^T = \mathbf{u} - \mathbf{u}_0$ and write the system of linearized variational equations in the form

$$\mathbf{B}(\mathbf{u}_0, M_0)\frac{d\Delta\mathbf{u}}{d\xi} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}_0, M_0)\Delta\mathbf{u}.$$

Eigenvalue problem which corresponds to this system reads

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}_0, M_0) - \lambda \mathbf{B}(\mathbf{u}_0, M_0)\right)\mathbf{r} = \mathbf{0}.$$

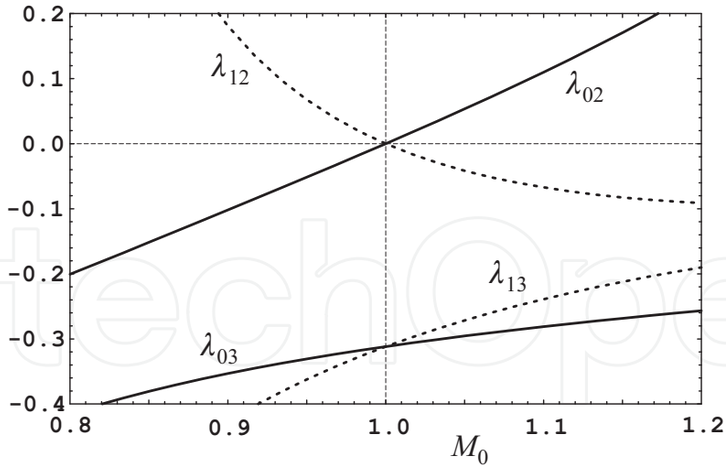

Fig. 2. Eigenvalues in upstream $\mathbf{u}_0$ and downstream $\mathbf{u}_1$ stationary points in He-Xe mixture shock structure.

Characteristic equation has generalized form

$$\det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}_0, M_0) - \lambda \mathbf{B}(\mathbf{u}_0, M_0)\right) = 0$$

and gives three eigenvalues

$$\lambda_{01} \equiv 0, \qquad \lambda_{02} = \lambda_{02}(M_0), \qquad \lambda_{03} = \lambda_{03}(M_0).$$

Note that one eigenvalue is identically zero, which is a consequence of the fact that among the system of shock structure equations (57) there is one conservation law. Other two eigenvalues can be calculated in a closed form, which is rather cumbersome. More important than its closed form is the fact that one of them changes the sign in the neighborhood of $M_0 = 1$

$$\lambda_{02}(1) = 0, \qquad \frac{d\lambda_{02}(1)}{dM_0} = 1.046; \qquad \lambda_{03}(1) = -0.3118, \tag{59}$$

the conclusion which can be drawn by continuity argument. By similar calculation one may determine the eigenvalues in downstream equilibrium state $\mathbf{u}_1$

$$\lambda_{11} \equiv 0, \qquad \lambda_{12} = \lambda_{12}(M_0), \qquad \lambda_{13} = \lambda_{13}(M_0).$$

Again, one of the eigenvalues is identically zero, and among other two there is one which changes the sign in the neighborhood of the critical value of shock speed $M_0 = 1$

$$\lambda_{12}(1) = 0, \qquad \frac{d\lambda_{12}(1)}{dM_0} = -1.046; \qquad \lambda_{13}(1) = -0.3118, \tag{60}$$

Figure 2 shows the graphs of nontrivial eigenvalues calculated in the mixture of Helium and Xenon, whose ratio of atomic masses is $m_{\mathrm{Xe}}/m_{\mathrm{He}} = 32.80$, and with equilibrium concentration of Helium $c_0 = 0.3$.

Arguments which relate the results in (59) and (60) to admissibility conditions are the same as in the case of gas dynamics. Namely, admissibility of the shock wave in equilibrium subsystem reads

$$\lambda_{\mathrm{E}}^{(5)}(\mathbf{u}_0) < s < \lambda_{\mathrm{E}}^{(5)}(\mathbf{u}_1).$$

Left inequality corresponds to $M_0 > 1$. For $s < \lambda_{\mathrm{E}}^{(5)}(\mathbf{u}_0)$, or $M_0 < 1$, shocks are not admissible. Moreover, $\lambda_{02}(M_0) < 0$ determines the stable direction (eigenvector) in state space when $M_0 < 1$, while $\lambda_{02}(M_0) > 0$ determines unstable one for $M_0 > 1$. The opposite conclusions are valid for $\lambda_{12}(M_0)$. Therefore, equivalently to (36), we can conclude that admissible shock structure appears when

$$0 < \lambda_{02}(M_0) \qquad (\lambda_{12}(M_0) < 0), \tag{61}$$

while $\lambda_{02}(M_0) = 0$, or $\lambda_{12}(M_0) = 0$, determine the critical value of the parameter (shock speed). Once again, the selection rule (61) is in direct relation to the properties of equilibrium subsystem, which are absorbed in the mixture model. Inequality form of admissibility criterion strengthens its irreversible nature. Finally, it has to be stressed that equilibrium states (stationary points $\mathbf{u}_0$ and $\mathbf{u}_1$), the critical value of the shock speed and stability condition (61) are determined from the shock structure equations (57) (or equivalently (52)) without prior solution of Rankine-Hugoniot equations and usage of Lax condition.

Explicit bifurcation analysis of the system (57), in the sense of previous two sections, will be omitted here due to huge computational efforts needed. However, from (58) and (55) it is obvious that $\lim_{M_0 \to 1} \mathbf{u}_1 = \mathbf{u}_0$ and that equilibrium states form two branches in the state space which intersect for $M_0 = 1$. This fact, along with existence of a simple eigenvalue which changes the sign, resembles the structure of transcritical bifurcation pattern discovered in previously studied models.

6. Conclusions

This study was concerned with stability and bifurcation analysis of equilibrium states in hyperbolic models of dissipative thermomechanical systems. It was focused on stability properties of equilibrium states connected by the shock structure. In particular, it was shown that the critical value of the speed of a travelling shock profile in dissipative system coincides with the critical value of the shock speed in equilibrium subsystem, the one which separates admissible from inadmissible shock waves according to Lax condition. The main feature of this analysis is its independence of previous study of equilibrium subsystem: the stationary points and their stability properties reflect the structure of equilibrium subsystem contained in dissipative one, but are determined without prior knowledge of solutions of Rankine-Hugoniot relations or application of Lax condition. The stability results for the models of isothermal viscoelasticity and gas dynamics, as well as the new ones for the multi-temperature model of mixture of Euler fluids, seem to be persuasive enough that we may conjecture that exchange of stability properties of stationary points may be used as a selection rule for admissible shock structures in dissipative hyperbolic systems.

Stability results shown in this chapter may be generalized and exploited in several ways. First, it is expected that stability and bifurcation of equilibrium can be proved for a general hyperbolic system of balance laws, under reasonable physical and structural assumption. This is a natural step towards their complete verification. Another possibility for application is numerical calculation of shock profiles. Eigenvectors corresponding to eigenvalues with changing sign determine asymptotic behavior of heteroclinic orbits in the neighborhood of stationary points when $\xi \to \pm\infty$. This fact can be used in truncation of the domain, necessary

for numerical computation of the profile, and determination of initial data for solving the problem as initial value one.

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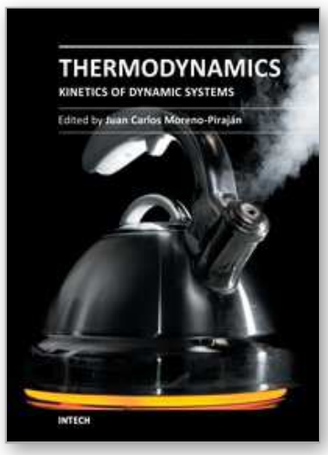

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